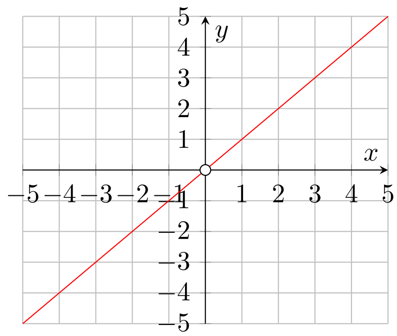
Calc I: Worksheet 1

Name: ____________________________

1. Consider the graph below of the function $f(x)$:



What is

(a) $\lim_{x\to 0^+} f(x)$?

(b) $\lim_{x\to 0^-} f(x)$?

(c) What can we say about $f(0)$?

(d) If the only hole in the graph is at $x = 0$, can you think of what the function $f(x)$ may be?

Hint: Think rational functions.

Solutions:

(a) $\lim_{x\to 0^+} f(x) = 0$. The function approaches $x = 0$ from the right-hand side via the line $y = x$ up until the value $x = 0$, as can be seen from the graph.

(b) $\lim_{x\to 0^-} f(x) = 0$. The function approaches $x = 0$ from the left-hand side via the line $y = x$ up until the value $x = 0$.

(c) The function is not defined at $x = 0$, which is indicated on the graph by the hole at the origin.

(d) The way to approach this problem is to try to remember examples where holes come about in graphs. This will happen with rational functions. Consider the following:

$$f(x) = \frac{x^2 - 4}{x - 2}$$

One might think at first that there is a vertical asymptote at $x = 2$, but by factoring the denominator, we see that

$$\begin{aligned} f(x) &= \frac{(x-2)(x+2)}{x-2} \\ &= (x+2)\left(\frac{x-2}{x-2}\right) \\ &= x+2, \ x \neq 2 \end{aligned}$$

so the graph looks just like $x+2$, except we must exclude $x = 2$ from the domain since it leads to division by zero. This means that the graph is $f(x) = x+2$ with *hole* at $x = 2$. The way to look at this is on the second line of the previous set of equations: we are multiplying the function $x+2$ by $\frac{x-2}{x-2}$, and this multiplication produces a hole at $x = 2$ in the graph of $f(x) = x + 2$.

This is completely analogous to the problem we're seeing here. Note that our graph looks like $f(x) = x$, with a hole at $x = 0$. So based on what we saw before, the way to generate this is to multiply x by $\frac{x}{x}$ to get our function:

$$g(x) = x\left(\frac{x}{x}\right) = \frac{x^2}{x}$$

Double check this answer to see that it in fact produces the graph we see.

2. Section 1.3, Q33:

(Sorry, I had to have at least one example of ϵ, δ theory here)!

Prove the statement using the ϵ, δ definition of the limit:

$$\lim_{x \to 1} \frac{2 + 4x}{3} = 2$$

Solution: The way to tackle ϵ, δ problems is to use the formal definition of the limit: Say our limit of the function $f(x)$ as $x \to a$ is truly L (which is what we're seeking to prove). Then if I pick any $\epsilon > 0$, I can guarantee the existence of some other positive number δ such that

$$|x - a| < \delta \implies |f(x) - L| < \epsilon$$

Let's look at our equation to identify what each of the terms are in our definition:

- The limit L is 2. We want to prove this is true.
- x is approaching 1, so $a = 1$
- $f(x) = \frac{2+4x}{3}$.
- We need to find δ based on whatever ϵ is.

What do we know about ϵ? We know that the absolute value of the difference between $f(x)$ and L as $x \to 1$ has to be less than it:

$$\lim_{x\to 1} \left| \frac{2+4x}{3} - 2 \right| < \epsilon$$

The goal now is to take the expression on the left-hand side and rearrange it to look like the expression

$$|x - 1| < \text{ something}$$

and that "something" will be what we end up choosing as our δ. These problems are often set up to be quite nice, as is this case. Let's work through it:

Let's start:

$$\begin{aligned}
\lim_{x\to 1} |2 + 4x - 6| &< 3\epsilon \\
\lim_{x\to 1} |-4 + 4x| &< 3\epsilon \\
\lim_{x\to 1} 4\,|x - 1| &< 3\epsilon \\
\lim_{x\to 1} |x - 1| &< \frac{3}{4}\epsilon
\end{aligned}$$

So we have arrived at the expression on the left-hand side that we needed! Thus, all we need to do is choose $\delta = \frac{3}{4}\epsilon$. Then

$$|x - 1| < \delta \implies \left| \frac{2+4x}{3} - 2 \right| < \epsilon$$

and we have proved that $x = 2$ is indeed the limit.

3. (a) Does $\lim_{x\to 0} x$ exist? If so, what is the limit?

   (b) Does $\lim_{x\to 0} |x|$ exist? If so, what is the limit?

   (c) Does $\lim_{x\to 0} \frac{|x|}{x}$ exist? If so, what is the limit?

   Solutions:

   (a) Yes, the limit exists and is zero.

(b) Yes, the limit exists and is zero.

(c) No! The right-hand limit approaches 1 and the left-hand limit approaches -1, so the limit does not exist. This is equivalent to the sign function, which is 1 if the input is positive and -1 if it is negative.

4. Compute the following limits:

   (a) $\lim_{x\to 0} \frac{\sin(x)}{x}$

   (b) $\lim_{x\to 0^+} \frac{\sin(x)}{\sqrt{x}}$

   (c) $\lim_{x\to 0} \frac{\sin(x)}{x^2}$

   Solutions:

   (a) $\lim_{x\to 0} \frac{\sin(x)}{x} = 1$, which you should already know!

   (b) $\lim_{x\to 0} \frac{\sin(x)}{\sqrt{x}} = \lim_{x\to 0} \sqrt{x}\frac{\sin(x)}{x} = \lim_{x\to 0} \sqrt{x} \lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \sqrt{x} = 0$

   (c) $\lim_{x\to 0} \frac{\sin(x)}{x^2} = \lim_{x\to 0} \frac{1}{x}\frac{\sin(x)}{x} = \lim_{x\to 0} \frac{1}{x} \lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{1}{x} = \text{DNE}$

5. Section 1.4, Q19

   Evaluate the limit, if it exists:

$$\lim_{x\to -2} \frac{x+2}{x^3+8}$$

   Solution: Let's work through it

$$\begin{aligned}
\lim_{x\to -2} \frac{x+2}{x^3+8} &= \lim_{x\to -2} \frac{x+2}{(x+2)(x^2-2x+4)} \\
&= \lim_{x\to -2} \frac{1}{x^2-2x+4} \\
&= \frac{1}{(-2)^2-2(-2)+4} \\
&= \frac{1}{4+4+4} \\
&= \frac{1}{12}
\end{aligned}$$

6. Section 1.4, Q22

   Evaluate the limit, if it exists:

$$\lim_{u\to 2} \frac{\sqrt{4u+1}-3}{u-2}$$

Solution: This requires the method of multiplying by the square root conjugate! The square root conjugate of the numerator is $\sqrt{4u+1}+3$, so multiply top and bottom by that:

$$\begin{aligned}
\lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2} &= \lim_{u\to 2}\frac{\sqrt{4u+1}-3}{u-2}\left(\frac{\sqrt{4u+1}+3}{\sqrt{4u+1}+3}\right)\\
&= \lim_{u\to 2}\frac{(\sqrt{4u+1}-3)(\sqrt{4u+1}+3)}{(u-2)(\sqrt{4u+1}+3)}\\
&= \lim_{u\to 2}\frac{4u+1-9}{(u-2)(\sqrt{4u+1}+3)}\\
&= \lim_{u\to 2}\frac{4u-8}{(u-2)(\sqrt{4u+1}+3)}\\
&= \lim_{u\to 2}\frac{4(u-2)}{(u-2)(\sqrt{4u+1}+3)}\\
&= \lim_{u\to 2}\frac{4}{\sqrt{4u+1}+3}\\
&= \frac{4}{\sqrt{4(2)+1}+3}\\
&= \frac{4}{3+3}\\
&= \frac{2}{3}
\end{aligned}$$

7. Section 1.4, Q35

   Prove that

$$\lim_{x\to 0} x^4\cos\left(\frac{2}{x}\right) = 0$$

   Solution: This is an example of the Squeeze Theorem. We are going to want to bound this function between two other functions that converge to the same limit, to show that the function too converges to that limit.

   We know that cosines are bounded between -1 and 1, so $\left|\cos\left(\frac{2}{x}\right)\right| \leq 1$ which tells us right away that

$$-|x^4| \leq x^4 \leq |x^4|$$

   and we know that as $x \to 0$, $-|x^4| \to 0$ and $|x^4| \to 0$. Thus, so does x^4.

8. Section 1.4, Q51

   Find the limit.

$$\lim_{t\to 0}\frac{\tan(6t)}{\sin(2t)}$$

Solution:

$$\begin{aligned}
\lim_{t\to 0}\frac{\tan(6t)}{\sin(2t)} &= \lim_{t\to 0}\frac{\sin(6t)}{\cos(6t)\sin(2t)} \\
&= \lim_{t\to 0}\frac{1}{\cos(6t)}\left(\frac{\sin(6t)}{t}\right)\frac{t}{\sin(2t)} \\
&= \lim_{t\to 0}\frac{1}{\cos(6t)}\left(\frac{\sin(6t)}{t}\right)\left(\frac{\sin(2t)}{t}\right)^{-1} \\
&= \lim_{t\to 0}\left(\frac{\sin(6t)}{t}\right)\left(\frac{\sin(2t)}{t}\right)^{-1} \\
&= 6(2)^{-1} \\
&= \frac{6}{2} = 3
\end{aligned}$$

9. Section 1.4, Questions 63 and 64

   (a) Show by means of an example that $\lim_{x\to a}[f(x)+g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

   (b) Show by means of an example that $\lim_{x\to a}[f(x)g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

Solution:

(a) Let

$$f(x) = \begin{cases} x+2 & x \le 0 \\ 0 & x > 0 \end{cases}$$

$$g(x) = \begin{cases} 0 & x < 0 \\ x+2 & x \ge 0 \end{cases}$$

Then individually, $f(x)$ and $g(x)$ have different left and right hand side limits, so their limits do not exist. But the sum $f(x)+g(x)$ is a simple linear polynomial, so the limits exist everywhere

$$\begin{aligned}
f(x)+g(x) &= x+2 \\
\lim_{x\to 0} f(x)+g(x) &= 2
\end{aligned}$$

(b) Let's slightly modify our above functions:

$$f(x) = \begin{cases} x+2 & x < 0 \\ 0 & x \ge 0 \end{cases}$$

$$g(x) = \begin{cases} 0 & x \le 0 \\ x+2 & x > 0 \end{cases}$$

So now $f(x) = g(x) = 0$ at $x = 0$. Then their product will be 0 everywhere, so the limit as $x \to 0$ will exist.

10. Section 1.4, Q65

Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so, find the value of a and find the limit.

Solution: Let's try factoring the denominator first:

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2} = \frac{3x^2 + ax + a + 3}{(x+2)(x-1)}$$

So for the limit to exist, we need to get rid of the $x + 2$ term in the denominator by having it cancel with a factor in the numerator. The numerator is quadratic, so we want to look for numbers b, c, d such that:

$$b(x+2)(cx+d) = 3x^2 + ax + a + 3$$

This will allow us to find the value of a that makes this true. Let's expand:

$$\begin{aligned} b(x+2)(cx+d) &= b(cx^2 + (d+2c)x + 2d) \\ &= bcx^2 + (d+2c)bx + 2db = 3x^2 + ax + a + 3 \end{aligned}$$

We see that

$$\begin{aligned} bc &= 3 \\ (d+2c)b &= a \\ 2db &= a + 3 \end{aligned}$$

So

$$\begin{aligned} c &= \frac{3}{b} \\ \left(d + 2\frac{3}{b}\right) b &= a \\ bd + 6 &= a \implies b = \frac{a-6}{d} \\ 2d\frac{a-6}{d} &= a + 3 \\ 2(a-6) &= a + 3 \\ 2a - 12 &= a + 3 \\ a &= 15 \end{aligned}$$

So if $a = 15$, this will work. Let's now substitute this into our original expression:

$$\begin{aligned}
\lim_{x \to -2} \frac{3x^2 + 15x + 15 + 3}{x^2 + x - 2} &= \lim_{x \to -2} \frac{3(x^2 + 5x + 6)}{(x+2)(x-1)} \\
&= \lim_{x \to -2} \frac{3(x+2)(x+3)}{(x+2)(x-1)} \\
&= \lim_{x \to -2} \frac{3(x+3)}{x-1} \\
&= \frac{3(-2+3)}{-2-1} \\
&= \frac{3}{-3} \\
&= -1
\end{aligned}$$